

# Orbit based procedure for doublets of scalar fields and the emergence of triple kinks and other defects

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June 9, 2014

## Abstract

In this work we offer an approach to enlarge the number of exactly solvable models with two real scalar fields in (1+1)D. We build some new two-field models, and obtain their exact orbits and exact or numerical field configurations. It is noteworthy that a model presenting triple-kinks and double-flat-top lumps is among those new models.

## 1 Introduction

The most part of the natural physical systems can be studied by using linear differential equations, with their good properties like the superposition principle. However, in the last few decades there is a growing in deal with system which are intrinsically nonlinear, specially those systems that supports topological defects. In fact, topological structures play an import role in the development in several branches of physics, from condensed matter to high energy physics and cosmology [1]-[4]. In condensed matter, a recent and interesting example regarding topological defects is related with the study of magnetic domain wall in a nanowire, designed for the development of magnetic memory [5]. In high energy physics we may cite, for instance, the importance of defect structures in brane world scenarios, where we may interpret that we live in a domain-wall with 3+1 dimensions embedded in a 5-dimensional spacetime [6, 7]. In cosmology, topological defects may be related with phase transitions in the early Universe, such defects may have formed as the Universe cooled and various local and global symmetries were broken [2, 8].

In this work we focus in models for doublets of scalar fields. It is remarkable that, whenever this models have a potential with two or more degenerate minima, one can find topological solutions connecting them. For models with a single scalar field it is usual we arrive at kink-like solutions. However, when we deal with models with two scalar fields the vacua structure may be richer, and as a consequence, other kinds of defects are possible. The so-called BNRT model [9], for instance, has a vacua structure with four degenerate minima and the doublet of scalar field admits kink-like (topological) solutions for one of its components and lump-like (non-topological) to the other one. For the same model, one can find double-kinks and flat-top lumps, and also, there is a critical case where both components of the doublet are kink-like configurations [10, 11]. As we will see, in this paper we will arrive with models that possess very interesting vacua structures, engendering kinks, double-kinks and even triple-kinks configurations.

In fact, double and triple-kinks are particular case of the so-called multikink configurations [12]. The interest in deal with multikinks was, in part, motivated by the discovery of Peyrard and Kruskal [13]

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that a single kink becomes unstable when it moves in a discrete lattice at sufficiently large velocity, while multikinks remains stable. This effect is associated with the interaction between the kink and the radiation, and the resonances were already observed experimentally [14]. Some years ago, Champney and Kivshar [15] performed an analysis on the reasons of the appearance of multikinks in dispersive nonlinear systems. Furthermore, multikinks have applications in different areas of physics. For instance: in may study of mobility hysteresis in a damped driven commensurable chain of atoms [16]. In high energy physics, double-kinks are important to explain the split-brane mechanism in braneworld scenarios [17, 18].

Another motivation to the study of models with two scalar fields, its related with intersection of defects and the construction of networks of defects [19, 20]. This subject may find applications, for instance, in cosmology [2], in magnetic materials [21] and in the study pattern formation in condensed matter [22]. Essentially the construction of networks of defects is related with a symmetry with respect to some discrete group (*e.g.*  $Z_3$  or  $Z_2 \times Z_2$ ) acting in the vacua structure. Hence, we are motivated with the possibility of construct new models for doublets of scalar fields with different vacua structures.

Unfortunately, as a consequence of the nonlinearity, we face some troubles when we deal with these systems analytically. The framework can be simplified considerably for systems in  $1 + 1$  dimensions, in this case we may reduce the set of second-order differential equations to a set of first-order ones, using the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) procedure [23]. However, if there is more than one scalar field in the model, those first-order differential equations are still coupled and the difficulty of solving the problem is in general great. In fact, the trial and error method historically arose due to the inherent difficulty to get general methods for solving nonlinear differential equations. Rajaraman [24] introduced an approach of this nature for the treatment of coupled relativistic scalar field theories in  $1 + 1$  dimensions. His procedure was model independent and can be used for search solutions in arbitrary coupled scalar fields models in  $1 + 1$  dimensions. However, the method is convenient and profitable only in some particular, but important, cases. Some years later, Bazeia and collaborators applied the approach developed by Rajaraman to some important models [25, 26]. Some years ago, it has been noted that in the case of the coupled nonlinear first-order equations, one can obtain a differential equation relating both fields, and its solution lead to a general orbit connecting the vacua of the model [10, 27]. The number of exact models with two scalar fields have been enlarged a little through the so-called deformation approach [28, 29]. Recently, it was shown that one can go further by performing a deformation of the orbit equation [30]. On the other hand, at least partially, one can devise the general behavior of the topological solutions of a given nonlinear model by studying its vacuum structure and its orbits [31]. In the last reference, it was noticed that the appearance of double-kink and flat-top lumps, was a consequence of the passage of the orbit in the vicinity of a vacuum, before to go to another one. As we are going to see in this work, in one of the new models introduced here, this feature will give rise to the emergence of triple-kinks and a kind of double flat-top lump.

Despite of all advances mentioned in the last paragraph, the number of nonlinear systems with two interacting scalar fields that can be exactly solved is yet relatively small, mostly due to the difficulty in getting solutions of the orbit equations. In this work we introduce an approach in order to tackle with this kind of problem. As we are going to see, it will allow us to expand very much the number of systems with two nonlinearly interacting scalar fields, for which one can get access to an analytical expression of the orbit equation and, as a consequence, construct solitonic configurations.

The Lagrangian density for the case of two coupled scalar fields that we are going to work with is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + \partial_\mu \chi \partial^\mu \chi) - V(\phi, \chi), \quad (1)$$

whose Euler-Lagrange equation for static configurations are

$$\frac{d^2 \phi}{dx^2} = \frac{\partial V}{\partial \phi} \quad , \quad \frac{d^2 \chi}{dx^2} = \frac{\partial V}{\partial \chi}. \quad (2)$$

An interesting consequence arises if one considers a class of potentials that can be written in terms of a superpotential function  $W(\phi, \chi)$ , namely

$$V(\phi, \chi) = \frac{1}{2}(W_\phi^2 + W_\chi^2), \quad (3)$$

in such case we are able to get a first order formalism to solve the problem, in fact, it is easy to verify that the following equations share the same solutions of (2)

$$\frac{d\phi}{dx} = W_\phi \quad , \quad \frac{d\chi}{dx} = W_\chi. \quad (4)$$

In general, the above equations are coupled through nonlinear terms. So, the usual methods of linear algebra are not useful in this case. In order to turn the above system decoupled, we note that it is possible to combine both equations to get the so called orbit equation

$$\frac{d\phi}{d\chi} = \frac{W_\phi}{W_\chi}, \quad (5)$$

once we solve this equation, we are going to be able to write  $\phi(\chi)$  or  $\chi(\phi)$  and, then, eliminate one of the fields on (4).

## 2 The method

In this section we are going to present a method to solve orbit equations. For this we consider an implicit solution of orbit equation given by  $F(\phi, \chi) = c$ , where  $c$  is a constant. Now, let us differentiate  $F(\phi, \chi)$  to obtain

$$dF(\phi, \chi) = \frac{\partial F}{\partial \phi} d\phi + \frac{\partial F}{\partial \chi} d\chi = 0, \quad (6)$$

we will also consider the orbit equation rewritten as  $W_\chi d\phi - W_\phi d\chi = 0$ . Moreover, one can multiply this equation by an integrating factor  $H(\phi)$  in order to get

$$H(\phi)W_\chi d\phi - H(\phi)W_\phi d\chi = 0. \quad (7)$$

Now, imposing the equivalence between these last two equations we obtain

$$\frac{\partial F}{\partial \phi} = H(\phi)W_\chi \quad , \quad \frac{\partial F}{\partial \chi} = -H(\phi)W_\phi. \quad (8)$$

It is important to observe that, the above equation provides a necessary condition to get a number of new interesting models, as we are going to see below. Once we determine  $H(\phi)$ , it can be replaced in (8) and, then, by direct integration we can obtain a solution for the orbit equation. In order to ensure that  $dF(\phi, \chi)$  is an exact differential, the following constraint must be fulfilled

$$\frac{\partial^2 F}{\partial \phi \partial \chi} = \frac{\partial^2 F}{\partial \chi \partial \phi}. \quad (9)$$

This last equation, along with (8), provides a definition of  $H(\phi)$ . So, applying the above condition in (8), we are led to the following relation between the superpotential and the integrating factor

$$\frac{W_{\phi\phi} + W_{\chi\chi}}{W_\phi} = -\frac{d \ln H(\phi)}{d\phi}. \quad (10)$$

It can be observed that the right-hand side of the above equation depends only on  $\phi$ . Therefore, the left-hand side must be just a function of  $\phi$  too. Thus, one might establish the following condition of applicability of the method

$$\frac{W_{\phi\phi} + W_{\chi\chi}}{W_{\phi}} = f(\phi). \quad (11)$$

By using the above condition, we identify a class of superpotentials that could be studied with this formalism. In fact, a large amount of models already considered in the literature satisfies the condition (11) [9, 10, 20, 30]. Moreover, the last equation could be integrated to give the integrating factor

$$H(\phi) = e^{-\int d\phi f(\phi)}. \quad (12)$$

At this point, it is important to stress that the equation (11) establishes a condition which is necessary and sufficient to construct the orbit equations of the new models that we are going to present in the next. However, it is also important to note that it is still necessary to write one field as a function of the other one, in order to obtain analytical solutions, and this restricts the set of useful orbits.

In order to exemplify the procedure above described, let us consider the so called BNRT model [9]. In this case, the superpotential is given by

$$W(\phi, \chi) = \lambda \left( \frac{1}{3} \chi^3 - a^2 \chi \right) + \mu \chi \phi^2, \quad (13)$$

Now, by checking that the condition (11) is satisfied, one gets

$$\frac{W_{\phi\phi} + W_{\chi\chi}}{W_{\phi}} = \frac{1 + \lambda/\mu}{\phi} = f(\phi). \quad (14)$$

Then, the integrating factor can be determined by using equation (12)

$$H(\phi) = \exp \left( - \int d\phi \frac{1 + \lambda/\mu}{\phi} \right) = \phi^{-(1+\lambda/\mu)} \quad (15)$$

by inserting this result in (8) we arrive at

$$\frac{\partial F}{\partial \phi} = \frac{\lambda(\chi^2 - a^2) + \mu\phi^2}{\phi^{(1+\lambda/\mu)}} \quad , \quad \frac{\partial F}{\partial \chi} = -2\mu\chi\phi^{-\lambda/\mu}. \quad (16)$$

Finally, by direct integration, we obtain

$$F(\phi, \chi) = \mu(\chi^2 - a^2)\phi^{-\lambda/\mu} - \frac{\mu}{2 - \lambda/\mu} \phi^{2-\lambda/\mu} = c, \quad (17)$$

which is the implicit form of the solution for the orbit equation. It is interesting to note that the above solution is the same that was obtained in ref. [10, 11]. However, since the solutions for the above model had already been studied in the literature we will not discuss it here.

### 3 Generating new nonlinear models

In this section we are going to present a systematic procedure that enables us to obtain new nonlinear scalar field models that satisfies the condition (11) and, as a consequence, the orbit equation of such systems arises naturally from the exact differential method. Essentially the procedure that we are going to introduce here consists in getting the solution of the equation

$$W_{\phi\phi} + W_{\chi\chi} = f(\phi)W_{\phi}. \quad (18)$$

It is important to stress out that we are not interest in obtain a general solution for the above partial differential equation with some boundary condition, in this paper we are interest in construct simple solutions in a systematic way engendering physically interesting model for doublets of scalar fields.

### 3.1 Polynomial model I:

Let us introduce the procedure through a concrete example. Consider the following *ansatz* for the general form of the superpotential

$$W(\phi, \chi) = a_{30}\phi^3 + a_{31}\phi^2\chi + a_{32}\phi\chi^2 + a_{33}\chi^3 + a_{10}\phi + a_{11}\chi, \quad (19)$$

where the coefficients are arbitrary. Note that the above superpotential do not presents any term with fourth degree. However, the respective potential function  $V(\phi, \chi)$  will have it. Substituting the last expression in (18) we obtain

$$(2a_{31} + 6a_{33})\chi + (6a_{30} + 2a_{32})\phi = f(\phi)[a_{32}\chi^2 + 2a_{31}\phi\chi + 3a_{30}\phi^2 + a_{10}], \quad (20)$$

comparing the coefficients of the above equation with respect to the powers of  $\chi$ , one can conclude that  $a_{32} = 0$  and  $2a_{31} + 6a_{33} = 2 a_{31} \phi f(\phi)$ . The last equation provide us a structure for the function  $f(\phi)$ , namely

$$f(\phi) = \frac{1 + 3a_{33}/a_{31}}{\phi}, \quad (21)$$

replacing it in equation (20) we may obtain  $2a_{30}\phi^2 = (3a_{30}\phi^2 + a_{10})(1 + 3a_{33}/a_{31})$ . Comparing the coefficients with respect to the powers of  $\phi$ , we may obtain  $a_{10} = 0$ ,  $a_{31} = 3a_{33}$ , and consequently  $f(\phi) = 2/\phi$ . Note that the other coefficients  $(a_{11}, a_{30}, a_{31})$  remain free. Thus, the superpotential (19) may be rewritten as follows

$$W(\phi, \chi) = a_{30}\phi^3 + a_{31}\left(\phi^2\chi + \frac{1}{3}\chi^3\right) + a_{11}\chi. \quad (22)$$

This superpotential generalizes the so called BNRT model [9], note that we recover the BNRT case when  $a_{30} = 0$ . The corresponding potential is

$$V(\phi, \chi) = \frac{1}{2}[3a_{30}\phi^2 + 2a_{31}\phi\chi]^2 + \frac{1}{2}[a_{31}(\phi^2 + \chi^2) + a_{11}]^2.$$

Using the same approach already realized in the previous section, we may obtain the implicit solution for the orbit equation

$$F(\phi, \chi) = a_{31}\left(\phi - \frac{\chi^2}{\phi}\right) - \frac{a_{11}}{\phi} - 3a_{30}\chi = c. \quad (23)$$

Now, let us look for a solution of this model. In this case we will restrict ourselves to the identification  $a_{11} = -a_{31} = 1$ , and also, we identify  $a_{30} = -\beta/3$ . So, the superpotential function may be rewritten as follows

$$W(\phi, \chi) = \chi - \frac{1}{3}\chi^3 - \chi\phi^2 - \frac{\beta}{3}\phi^3, \quad (24)$$

The first order equation derived from this superpotential may be written as

$$\frac{d\phi}{dx} = -2\chi\phi - \beta\phi^2 \quad , \quad \frac{d\chi}{dx} = 1 - \chi^2 - \phi^2. \quad (25)$$

In fact, this superpotential is an asymmetric version of the so called BNRT model discussed above, and it is not difficult to see that the addition of the term  $\beta\phi^3/3$  in the superpotential, breaks the  $Z_2 \times Z_2$  symmetry that is presented in the potential of the BNRT model. The vacua of the model may be obtained, as usual, from  $W_\phi = W_\chi = 0$ . In this case we get four vacua corresponding to the coordinates  $(\phi_v, \chi_v)$  in the internal space

$$\begin{aligned} v_1 &= (0, 1) & v_2 &= \left(\frac{2}{\sqrt{\beta^2 + 4}}, \frac{-\beta}{\sqrt{\beta^2 + 4}}\right) & v_3 &= (0, -1) \\ v_4 &= \left(\frac{-2}{\sqrt{\beta^2 + 4}}, \frac{\beta}{\sqrt{\beta^2 + 4}}\right). \end{aligned} \quad (26)$$

By using equation (23) we can express the orbit equation as follows

$$\chi^2 - 1 + \beta\phi\chi = \phi^2 - b\sqrt{\beta^2 + 4}\phi, \quad (27)$$

where the identification  $c \equiv -b\sqrt{\beta^2 + 4}$  was used. It is interesting to note that the vacua states  $v_1$  and  $v_3$  always satisfy the above equation, independently of the values of  $b$ . However, the other two vacua, namely  $v_2$  and  $v_4$ , only satisfy (27) if the parameter  $b$  is taken to be equal to the following critical values  $b = \pm 1$ .

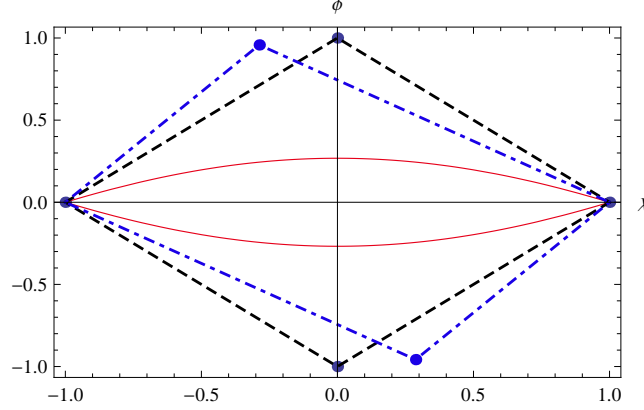


Figure 1: Vacua structure and orbits solutions.  $\beta = 0$  and  $b = 2$  - solid line (red);  $\beta = 0$  and  $b = 1,000000001$  - dashed line (black);  $\beta = 0,6$  and  $b = 1,000000001$  - dotdashed line (blue);

In order to decouple the pair of first order equations (25) one can use the orbit (27) to express  $\phi$  as a function of  $\chi$ , so that

$$\phi(\chi) = \frac{\beta\chi + b\sqrt{\beta^2 + 4} - f(\chi)}{2}, \quad (28)$$

where  $f(\chi) = \sqrt{(\chi\sqrt{\beta^2 + 4} + \beta b)^2 + 4(b^2 - 1)}$ . Substituting it in the second equation of (25) and performing the integration, we obtain the following solution

$$\chi(x) = \sqrt{\frac{b^2 - 1}{\beta^2 + 4}} \left[ \frac{2 \tanh(x - x_0) + 2b}{\sqrt{b^2 - 1}(\sqrt{\beta^2 + 4} - \beta)} - \frac{\sqrt{b^2 - 1}(\sqrt{\beta^2 + 4} - \beta)}{2 \tanh(x - x_0) + 2b} - \frac{\beta b}{\sqrt{b^2 - 1}} \right].$$

The other field,  $\phi(x)$ , can be obtained by direct substitution of the explicit form of  $\chi(x)$  in the equation (28). As one can see in Figure 2, the field  $\chi(x)$  presents a kink-like behavior while  $\phi(x)$ , exhibits a lump-like profile. It is a remarkable fact that when  $b = \pm(1 + \varepsilon)$  (with  $\varepsilon$  being a positive and very small parameter), the field  $\chi(x)$  develops a two-kink behavior, and  $\phi(x)$ , shows a flat-top region on the lump structure. In the exact situation with  $b = \pm 1$ , both fields present a kink-like structure. It is interesting to note that one can recover the results obtained in [27] by choosing  $\beta = 0$ .

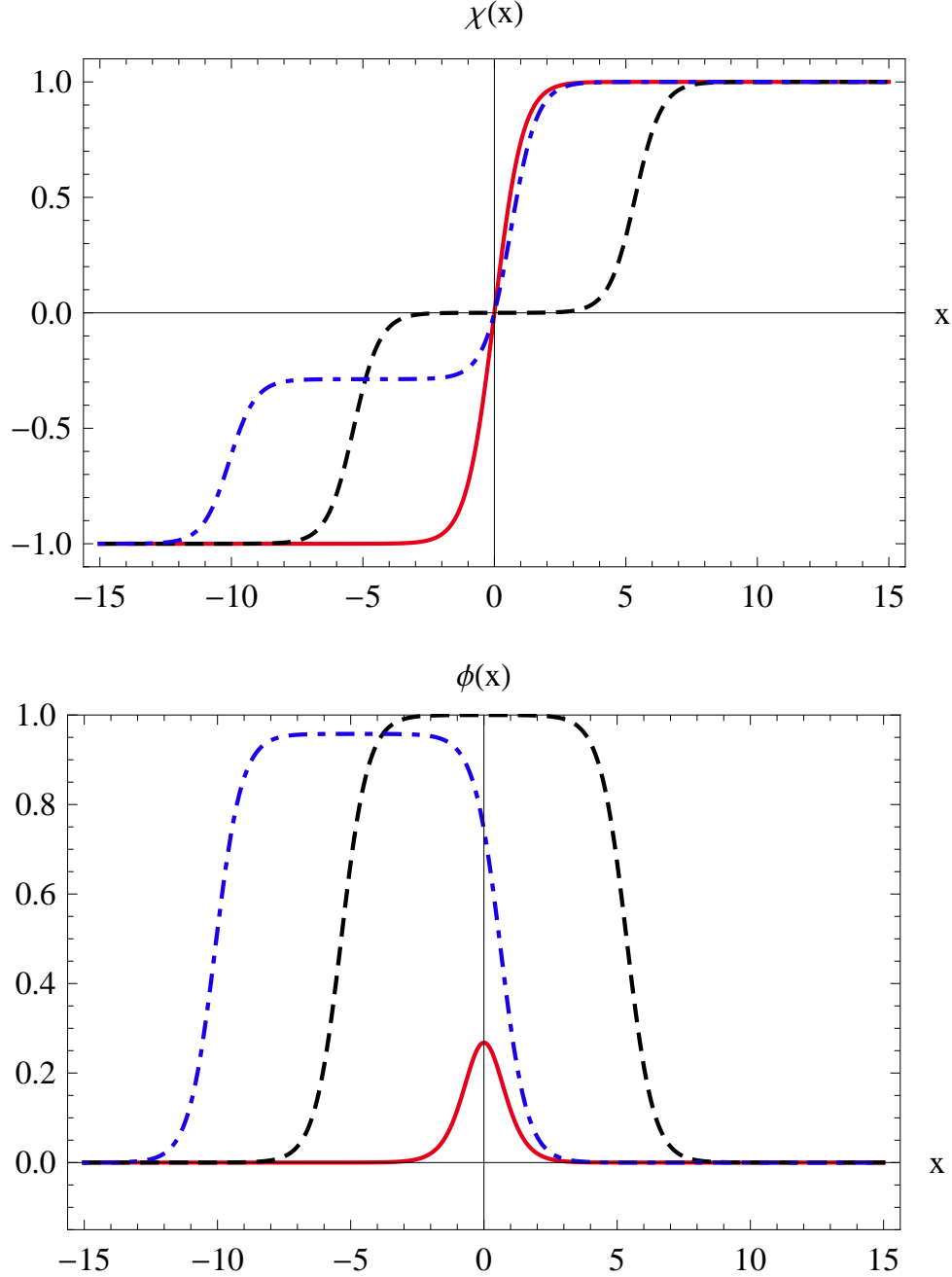


Figure 2: Double-kink (upper panel) and a flat-top lump.  $\beta = 0$  and  $b = 2$  - solid line (red);  $\beta = 0$  and  $b = 1,000000001$  - dashed line (black);  $\beta = 0,6$  and  $b = 1,000000001$  - dotdashed line (blue);

### 3.2 Polynomial model II:

The second model to be considered here is a polynomial superpotential with fourth power degree terms (the corresponding potential  $V(\phi, \chi)$  will contain terms with power of sixth degree in the fields). For this, we consider the following structure for the superpotential

$$W(\phi, \chi) = a_{40}\phi^4 + a_{41}\phi^3\chi + a_{42}\phi^2\chi^2 + a_{43}\phi\chi^3 + a_{44}\chi^4 + a_{20}\phi^2 + a_{21}\phi\chi + a_{22}\chi^2.$$

Repeating the same procedure used in the previous section we may conclude that  $f(\phi) = 2/\phi$  and consequently  $H(\phi) = 1/\phi^2$ . Adjusting the coefficients by the same method of the previous section and

then substituting it in the superpotential, we get

$$W(\phi, \chi) = a_{40} \left( \phi^4 - 2\phi^2\chi^2 - \frac{1}{3}\chi^4 \right) + a_{41}\phi^3\chi + a_{20}(\phi^2 + \chi^2), \quad (29)$$

and we get the following potential

$$V(\phi, \chi) = \frac{1}{2}[a_{40}(4\phi^3 - 4\phi\chi^2) + 3a_{41}\phi^2\chi + 2a_{20}\phi]^2 + \frac{1}{2}[a_{40}(-4\phi^2\chi - 4\chi^3/3) + a_{41}\phi^3 + 2a_{20}\chi]^2.$$

By replacing the expressions above obtained in (8), and integrating them in their respective variables, we may obtain that the implicit solution for the orbit equation that is given by

$$F(\phi, \chi) = a_{40} \left( \frac{4\chi^3}{3\phi} - 4\phi\chi \right) + \frac{a_{41}}{2}\phi^2 - 2a_{20}\frac{\chi}{\phi} - \frac{3a_{41}}{2}\chi^2 = c. \quad (30)$$

Now, we are going to look for solutions of this model. We will restrict ourselves to the case where  $a_{41} = 0$ , and also, us identify  $a_{40} = 1/4$  and  $a_{20} = \beta^2/6$ . Then the superpotential function can be rewritten as follow

$$W(\phi, \chi) = \frac{1}{4} \left( \phi^4 - 2\phi^2\chi^2 - \frac{1}{3}\chi^4 \right) + \frac{\beta^2}{6}(\phi^2 + \chi^2).$$

The corresponding first order differential equations are given by

$$\frac{d\phi}{dx} = \phi^3 - \phi\chi^2 + \frac{\beta^2}{3}\phi \quad , \quad \frac{d\chi}{dx} = -\phi^2\chi - \frac{1}{3}\chi^3 + \frac{\beta^2}{3}\chi, \quad (31)$$

and the orbit (30) turns out to be

$$\left( \frac{\beta^2 + 3\phi^2 - \chi^2}{3} \right) \chi - \frac{\beta^2 b}{\sqrt{3}} \phi = 0, \quad (32)$$

where the redefinition  $c \equiv -\beta^2 b/\sqrt{3}$  was used. The potential  $V(\phi, \chi)$  possess seven different vacua states that are given by

$$\begin{aligned} v_1 &= (0, 0) & v_2 &= \left(0, \sqrt{\beta}\right) & v_3 &= \left(0, -\sqrt{\beta}\right) & v_4 &= \left(\frac{\beta}{\sqrt{6}}, \frac{\beta}{\sqrt{2}}\right) \\ v_5 &= \left(-\frac{\beta}{\sqrt{6}}, -\frac{\beta}{\sqrt{2}}\right) & v_6 &= \left(-\frac{\beta}{\sqrt{6}}, \frac{\beta}{\sqrt{2}}\right) & v_7 &= \left(\frac{\beta}{\sqrt{6}}, -\frac{\beta}{\sqrt{2}}\right). \end{aligned} \quad (33)$$

In the Figure 3 we plot the vacua structure and some possible orbits.

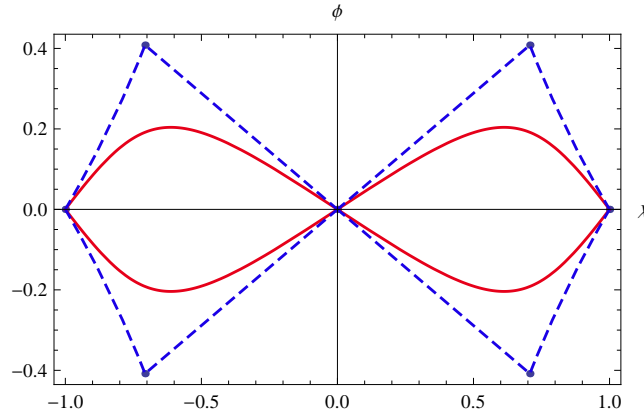


Figure 3: Vacua structure and orbits solutions.  $\beta = 1$  and  $b = 1, 3$  - solid line (red);  $\beta = 1$  and  $b = 1,00000001$  - dashed line (blue);



An interesting fact is that it is possible to obtain analytical solutions for this model, unlike other models with sixth degree terms on its potential [31]. Note that it is possible to use the above orbit in order to express  $\phi$  in terms of  $\chi$  and then substitute it in (31). Performing some changes of variables we may integrate the remaining first order equation to obtain

$$\chi(x) = \pm \frac{\beta}{2} \left[ 2 + b + \tanh(\beta^2(x - x_0)/3) - \frac{b^2 - 1}{\tanh(\beta^2(x - x_0)/3) + b} \right]^{1/2}, \quad (34)$$

the field  $\phi(x)$  can be determined by direct substitution of the last equation into the orbit equation, resulting into the following expression

$$\phi(\chi) = \frac{b\beta^2}{2\sqrt{3}\chi} \left[ 1 - \sqrt{1 + \frac{4\chi^2}{b^2\beta^4}(\chi^2 - \beta^2)} \right] \quad (35)$$

In the Figure 4 we plot the solutions mentioned above for some specific values of  $\beta$  and  $b$ , note that the field  $\chi(x)$  present an asymmetrical two-kink like profile when the integration constant is close to a certain critical value, while the other field  $\phi(x)$  exhibits a lump-like solution with a flat top region.

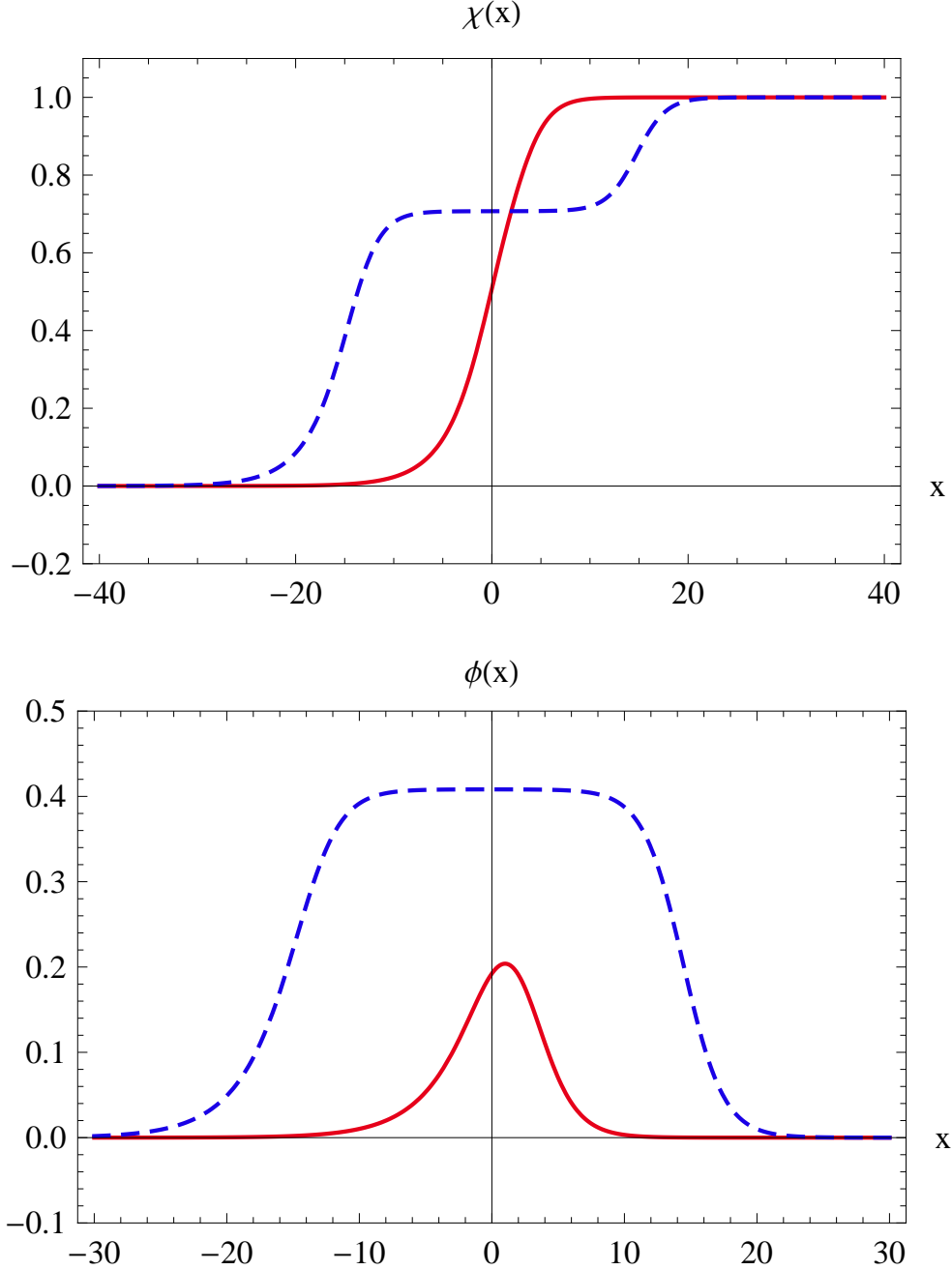


Figure 4: Fields solutions for the polynomial model II.  $\beta = 1$  and  $b = 1, 3$  - solid line (red);  $\beta = 1$  and  $b = 1, 00000001$  - dashed line (blue);

### 3.3 Polynomial model III:

The third model we consider is characterized by a polynomial superpotential containing terms with fifth power degree, whose corresponding potential  $V(\phi, \chi)$  contains eighth degree terms. Let us consider the following structure for the superpotential

$$\begin{aligned}
 W(\phi, \chi) = & a_{50}\phi^5 + a_{51}\phi^4\chi + a_{52}\phi^3\chi^2 + a_{53}\phi^2\chi^3 + a_{54}\phi\chi^4 + a_{55}\chi^5 + \\
 & + a_{30}\phi^3 + a_{31}\phi^2\chi + a_{32}\phi\chi^2 + a_{33}\chi^3 + a_{10}\phi + a_{11}\chi.
 \end{aligned} \tag{36}$$

Using the same procedure of the previous sections we obtain, once more, that  $f(\phi) = 2/\phi$  and, as a consequence,  $H(\phi) = 1/\phi^2$ . Adjusting the coefficients by the same method of the previous section and

then substituting it in the superpotential, one may conclude that

$$W(\phi, \chi) = a_{50}(\phi^5 - 5\phi^3\chi^2) + a_{31}\left(\phi^2\chi + \frac{1}{3}\chi^3\right) + a_{30}\phi^3 + a_{51}\left(\phi^4\chi - \frac{2}{3}\phi^2\chi^3 - \frac{1}{15}\chi^5\right) + a_{11}\chi.$$

The implicit solution for the orbit equation is

$$F(\phi, \chi) = 5a_{50}(\chi^3 - \phi^2\chi) + a_{51}\left(\frac{1}{3}\phi^3 - 2\phi\chi^2 + \frac{\chi^4}{3\phi}\right) + a_{31}\left(\phi - \frac{\chi^2}{\phi}\right) - 3a_{30}\chi - \frac{a_{11}}{\phi} = c.$$

Let us study the particular case with  $a_{50} = a_{30} = 0$ ,  $a_{51} = 1$ ,  $a_{31} = \gamma$  and  $a_{11} = \beta/4$  (with  $\gamma$  and  $\beta$  positives). Unfortunately, in this case, it will be possible to carry out analytical calculations only partially. However, it is interesting to analyze this model since some interesting features will arise. Also, this case is a good example to show the importance of obtaining an analytical expression for the orbit solution. Substituting the specific values of the coefficients in the superpotential function we get

$$W(\phi, \chi) = \phi^4\chi - \frac{2}{3}\phi^2\chi^3 - \frac{\chi^5}{15} + \gamma\left(\phi^2\chi + \frac{1}{3}\chi^3\right) + \frac{\beta}{4}\chi. \quad (37)$$

The corresponding first order differential equations are given by

$$\frac{d\phi}{dx} = 4\phi^3\chi - \frac{4}{3}\phi\chi^3 + 2\gamma\phi\chi \quad , \quad \frac{d\chi}{dx} = \phi^4 - 2\phi^2\chi^2 - \frac{1}{3}\chi^4 + \gamma(\phi^2 + \chi^2) + \frac{\beta}{4}, \quad (38)$$

and the orbit solution is

$$\frac{1}{3}(\phi^4 - 6\phi^2\chi^2 + \chi^4) + \gamma(\phi^2 - \chi^2) - \frac{\beta}{4} = c\phi \quad (39)$$

The potential function  $V(\phi, \chi)$  possess six vacua. In order to specify this vacua let us define the following quantities

$$\begin{aligned} \phi_{vac}^{(1)} &= 0 & \phi_{vac}^{(2)} &= \left[-\frac{\gamma}{8} + \frac{\sqrt{2\beta + 7\gamma^2}}{2}\right]^{1/2} & \chi_{vac}^{(1)} &= \left[\frac{3\gamma + \sqrt{3\beta + 9\gamma^2}}{2}\right]^{1/2} \\ \chi_{vac}^{(2)} &= \left[\frac{9\gamma + 3\sqrt{2\beta + 7\gamma^2}}{8}\right]^{1/2} \end{aligned} \quad (40)$$

thus, the corresponding coordinates of the vacua states in the internal space may be written as follows

$$\begin{aligned} v_1 &= (\phi_{vac}^{(1)}, \chi_{vac}^{(1)}) & v_2 &= (\phi_{vac}^{(1)}, -\chi_{vac}^{(1)}) & v_3 &= (\phi_{vac}^{(2)}, \chi_{vac}^{(2)}) \\ v_4 &= (\phi_{vac}^{(2)}, -\chi_{vac}^{(2)}) & v_5 &= (-\phi_{vac}^{(2)}, \chi_{vac}^{(2)}) & v_6 &= (-\phi_{vac}^{(2)}, -\chi_{vac}^{(2)}). \end{aligned} \quad (41)$$

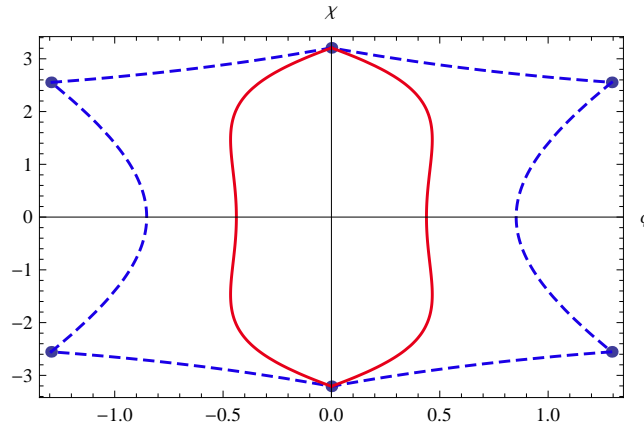


Figure 5: Vacua structure and orbits solutions.  $\beta = 100$ ,  $\gamma = 1$  and  $b = 2$  - solid line (red);  $\beta = 100$ ,  $\gamma = 1$  and  $b = 1 + 10^{-40}$  - dashed line (blue);

It is not difficult to verify that the vacua states  $v_1$  and  $v_2$  satisfy (39) independently of the value chosen for the constant  $c$ . Otherwise, other vacua states satisfies the orbit (39) only for some critical value of  $c$  that is determined by the substitution of coordinates of the vacua sates in the orbit solution [31]. For instance, let us consider the vacua  $v_3$ , the critical value  $c_0$  is given by

$$c_0 = \frac{1}{\phi_{vac}^{(2)}} \left[ \frac{1}{3} \left( (\phi_{vac}^{(2)})^4 - 6(\phi_{vac}^{(2)})^2 (\chi_{vac}^{(2)})^2 + (\chi_{vac}^{(2)})^4 \right) + \gamma \left( (\phi_{vac}^{(2)})^2 - (\chi_{vac}^{(2)})^2 \right) - \frac{\beta}{4} \right]. \quad (42)$$

It is easy to see that the vacuum  $v_4$  possess the same value for critical constant while  $v_5$  and  $v_6$  possess the critical value  $-c_0$ . It is interesting rewrite the orbit solution in terms of this critical parameter, namely

$$\frac{1}{3}(\phi^4 - 6\phi^2\chi^2 + \chi^4) + \gamma(\phi^2 - \chi^2) - \frac{\beta}{4} = b \ c_0 \ \phi. \quad (43)$$

While  $c$  defines a family of orbits in equation (39),  $b$  defines a family of orbits in the above equation. In the Figure 5 we plot the orbit solution for some values of the parameters  $b$ ,  $\gamma$  and  $\beta$ . Finally, in order to perform the numerical integration in the first order equation (38) we have to specify initial values for both fields, and the orbit solution is very useful at this point. For instance, let us look for solutions that connect  $v_1$  and  $v_2$ , certainly there exists a point  $x_0$  in which  $\chi(x_0) = 0$  (we will choose  $x_0 = 0$  without loosing generality, since the problem possess a translational invariance). The corresponding value of  $\phi$  at  $x = 0$  can be directly determined from the orbit solution, by solving the following equation

$$\frac{1}{3}\phi(0)^4 + \gamma\phi(0)^2 - bc\phi(0) - \frac{\beta}{4} = 0. \quad (44)$$

In the Figure 6 we plot the numerical solution obtained through this procedure for some values of the parameters  $b$ ,  $\gamma$  and  $\beta$ . Note that, for some values of  $b$ , the field  $\chi(x)$  presents a kind of triple kink configuration while the field  $\phi(x)$  present double lump behavior with a flat-top region. In the Figure 7 one can see that there exists three regions with a formation of peaks in the energy density. As far we know this kind of configuration was never presented in the literature. In fact, very recently a solution like those was obtained in a model with one self-interacting scalar field [12]. However, beyond the fact that there is only one field in the model, the potential is not entirely continuous, instead, it is continuous by parts. Here, the model is absolutely continuous and the model is for a doublet of scalar fields.

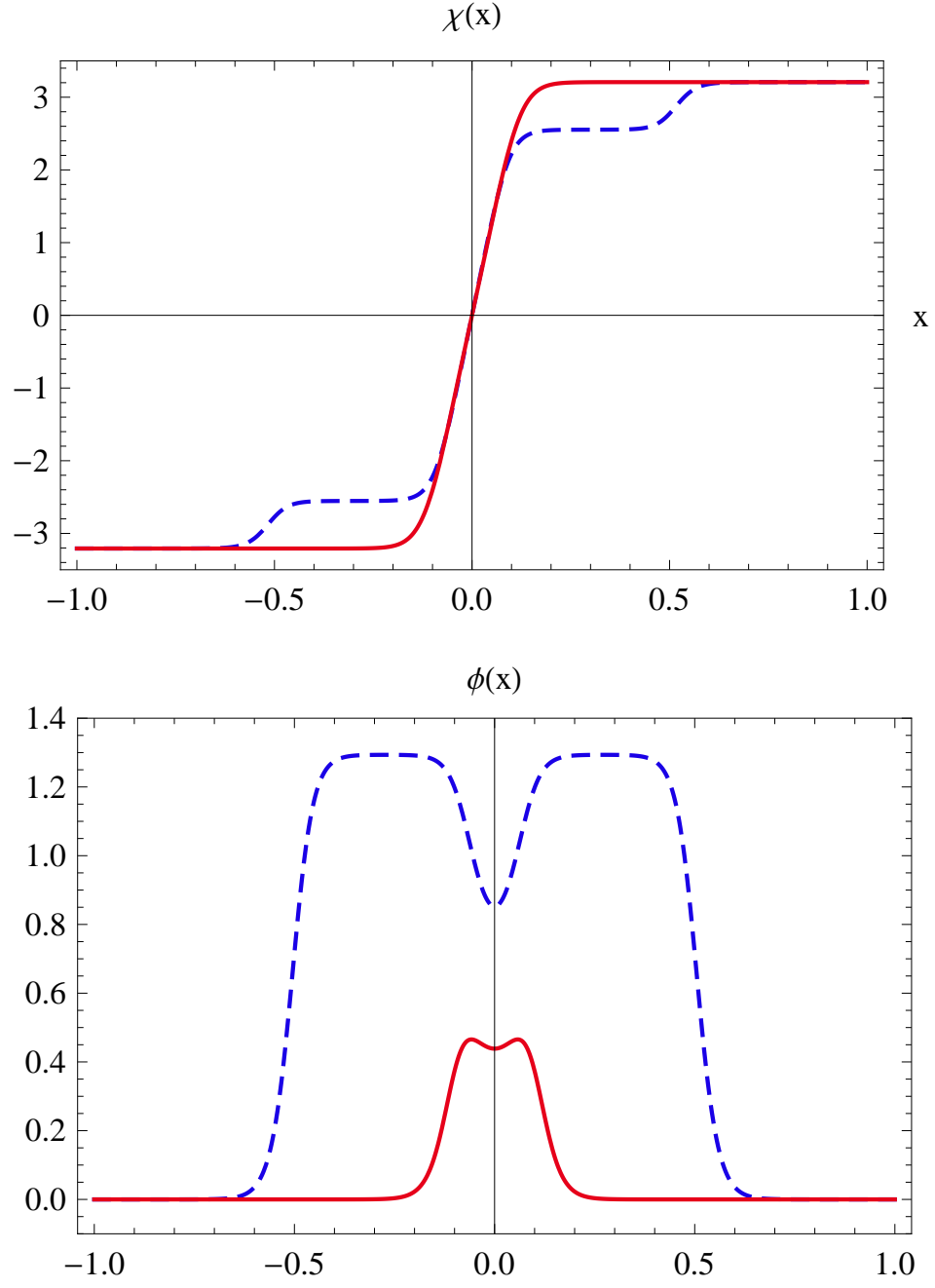


Figure 6: Fields solutions for the polynomial model III.  $\beta = 100$ ,  $\gamma = 1$  and  $b = 2$  - solid line (red);  $\beta = 100$ ,  $\gamma = 1$  and  $b = 1 + 10^{-40}$  - dashed line (blue);

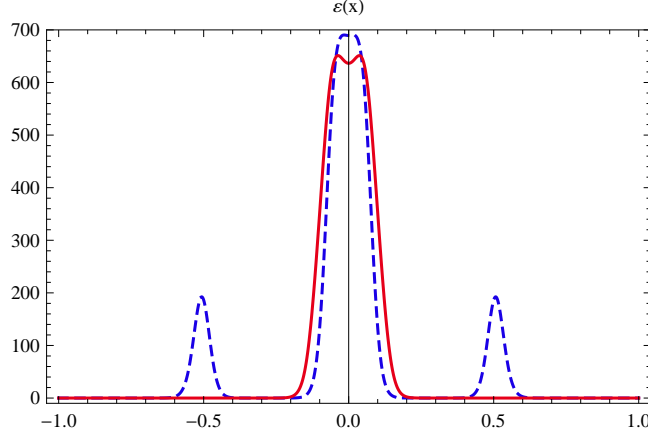


Figure 7: Energy density for the polynomial model III.  $\beta = 100$ ,  $\gamma = 1$  and  $b = 2$  - solid line (red);  $\beta = 100$ ,  $\gamma = 1$  and  $b = 1 + 10^{-40}$  - dashed line (blue);

### 3.4 Generalized polynomial model:

In this section we are going to generalize the procedure exemplified in the preceding sections and obtain a polynomial superpotential with Nth degree terms. For this we consider the following superpotential

$$W^{(N)}(\phi, \chi) = \sum_{n=0}^N W^{(n)}(\phi, \chi), \quad (45)$$

where

$$W^{(n)}(\phi, \chi) = \sum_{l=0}^n a_{nl} \phi^{n-l} \chi^l. \quad (46)$$

Note that the equation  $W_{\phi\phi}^{(N)} + W_{\chi\chi}^{(N)} = f(\phi)W_{\phi}^{(N)}$  is linear in  $W^{(N)}$ . Therefore, one can solve the equations  $W_{\phi\phi}^{(n)} + W_{\chi\chi}^{(n)} = f(\phi)W_{\phi}^{(n)}$  individually and, then, sum over its solutions in order to obtain  $W^{(N)}$ . The three models considered previously in this work provide us the result  $f(\phi) = 2/\phi$ . Thus, it seems interesting to consider this result in this generalization. From now on, our task reduce to solve the following equation

$$W_{\phi\phi}^{(n)} + W_{\chi\chi}^{(n)} = \frac{2}{\phi} W_{\phi}^{(n)}. \quad (47)$$

Substituting the superpotential (46) in the above equation, and repeating the same procedure of the previous sections, in other words, comparing the coefficients accordingly to the degree of  $\phi$  and  $\chi$ , we may obtain  $a_{n(n-1)} = 0$  and also the following recurrence formula

$$a_{n(l+2)} = \frac{(n-l)(3+l-n)}{(l+2)(l+1)} a_{nl}. \quad (48)$$

by successive applications of the above recurrence relation, we may find that the general term is given by

$$a_{nl} = \frac{n!!}{l!(n-l)!!} \prod_{k=0}^{(l-2)/2} (3+2k-n) a_{n0}, \quad (49)$$

for  $l$  even, and

$$a_{nl} = \frac{(n-1)!!}{l!(n-l)!!} \prod_{k=0}^{(l-3)/2} (4+2k-n) a_{n1}, \quad (50)$$

for  $l$  odd. Above we have used  $n!! = n(n-2)!!$ .

Now we turn our attention to the solution of the orbit equation, which is in general nonlinear in terms of the fields. However, it is linear in terms of the implicit solution  $F(\phi, \chi)$ . Therefore, we may look for solutions in the form

$$F(\phi, \chi) = \sum_{n=0}^N F^{(n)}(\phi, \chi) = c, \quad (51)$$

where the function  $F^{(n)}(\phi, \chi)$  satisfies the following equation

$$dF^{(n)}(\phi, \chi) = \frac{\partial F^{(n)}}{\partial \phi} d\phi + \frac{\partial F^{(n)}}{\partial \chi} d\chi = H(\phi)W_\chi^{(n)} d\phi - H(\phi)W_\phi^{(n)} d\chi = 0.$$

By comparison of the terms in the above equation, we find that

$$\frac{\partial F^{(n)}}{\partial \phi} = H(\phi)W_\chi^{(n)}, \quad \frac{\partial F^{(n)}}{\partial \chi} = -H(\phi)W_\phi^{(n)}. \quad (52)$$

Integrating the above equations and taking into account the recurrence relation (48), we may obtain the following result

$$F^{(n)}(\phi, \chi) = \sum_{l=1}^n \frac{l}{n-l-1} a_{nl} \phi^{n-l-1} \chi^{l-1} + \beta^{(n)}(\chi) = c_n, \quad (53)$$

where

$$\beta^{(n)}(\chi) = \begin{cases} 0 & , n < 3 \\ \frac{3}{2-n} a_{n(n-3)} \chi^{n-2} & , n \geq 3. \end{cases} \quad (54)$$

summing over all the possible values of  $n$ , we get

$$F(\phi, \chi) = \sum_{n=1}^N \sum_{l=1}^n \frac{l}{n-l-1} a_{nl} \phi^{n-l-1} \chi^{l-1} + \sum_{n=1}^N \beta^{(n)}(\chi) = c.$$

Note that

$$\sum_{n=1}^N \beta^{(n)} = \sum_{n=3}^N \frac{3}{2-n} a_{n(n-3)} \chi^{n-2}, \quad (55)$$

thus, we obtain

$$F(\phi, \chi) = \sum_{n=1}^N \sum_{l=1}^n \frac{l}{n-l-1} a_{nl} \phi^{n-l-1} \chi^{l-1} + \sum_{n=3}^N \frac{3}{2-n} a_{n(n-3)} \chi^{n-2} = c.$$

Naturally, we cannot obtain an analytical solution for the above model. However, as it was pointed out in the previous section, the knowledge of an analytical expression for the orbit is an important step for the analysis of nonlinear scalar field theories. This happens because it allows one to decouple the first order differential equations and choosing adequately boundary conditions.

### 3.5 Nonlinear oscillating models

The systematic procedure developed in the last sections with polynomial models, can be extended to build up models with potentials presenting harmonic functions of the fields. For instance, let us consider the following *ansatz* for an oscillating superpotential

$$W(\phi, \chi) = A \sin \phi \sin \chi + B \cos \phi \cos \chi + C \sin \phi \cos \chi + D \cos \phi \sin \chi + E\phi + F\chi.$$

Following the same procedure of the previous sections, one can substitute the above superpotential in (18), and by comparing the involved terms, one may obtains that

$$f(\phi) = \frac{2(A \sin \phi + D \cos \phi)}{D \sin \phi - A \cos \phi}, \quad (56)$$

where the coefficients must obey the following constraints:  $AB = CD$ ,  $E = 0$ . In the case where  $C \neq 0$  we get  $D = AB/C$ . Thus, the superpotential can be rewritten as

$$W(\phi, \chi) = A(\sin \chi + C \cos \chi)(\sin \phi + B/C \cos \phi) + F\chi. \quad (57)$$

The integrating factor obtained by using  $f(\phi)$  is given by

$$H(\phi) = \frac{1}{A^2(B/C \sin \phi - \cos \phi)^2}, \quad (58)$$

consequently, we get the following orbit

$$F(\phi, \chi) = \frac{C \sin \chi - A \cos \chi}{A^2(B/C \sin \phi - \cos \phi)} + \frac{F/A^2}{B/C(1 - B/C \tan^2 \phi)} = c.$$

As far as we know, this superpotential was not considered in the literature. However, one can note that in the case where  $C = 0$  we may recover the oscillating model studied in ref. [30].

## 4 Conclusions

In this work, we introduced a method which allows one obtain the solutions of the orbit equation for the case of nonlinearly coupled two scalar fields and, beyond that, we present a procedure that allows the construction of new exact nonlinear models of this nature systematically, in such a way that the solution of the orbit equation appears naturally. By applying the method we have studied, some novel polynomial models were introduced and we explored the behavior of their solitonic configurations. We have also verified that this procedure can be extended to the case of oscillating potentials.

In the first model analyzed, we got a generalization of the BNRT [9] model, which presents a structure with four vacua and a parameter which controls the asymmetry of the position of those vacua. It is noteworthy that this model presents important consequences in the braneworld scenario [17]. The second model which was analyzed here, was one of the sixth degree in the fields and in analytically solvable in contrast with happens with some others in the literature [31]. It is remarkable that the third polynomial potential introduced in this work, exemplifying of the powerness of the approach, presents configurations with a triple kink and a kind of double-flat-top lump.

**Acknowledgements:** The authors thanks to CNPq and FAPESP for partial financial support.

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